SPLITTING THE SHADOW

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RÉSUMÉ :
Nous évaluons les séries thetas de deux moitiés de l’ombre d un réseau arithmétique impair. Ce résultat est généralisé aux anneaux et est utilisé ensuite pour construire des codes formellement auto dually et des empilements de sphères.

MOTS CLÉS :

ABSTRACT:
We derive formulae for the theta series of the two translates of the even sublattice $L_0$ of an odd unimodular lattice $L$ that constitute the shadow of $L$. The proof rests on special evaluations of the Jacobi theta series attached to $L$ and to a certain vector. We produce an analogous theorem for codes. Additionally, we construct non-linear formally self-dual codes and relate them to lattices.

KEY WORDS:
shadow lattice, Jacobi modular forms
Splitting the Shadow

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Abstract

We derive formulae for the theta series of the two translates of the even sublattice \( L_0 \) of an odd unimodular lattice \( L \) that constitute the shadow of \( L \). The proof rests on special evaluations of the Jacobi theta series attached to \( L \) and to a certain vector. We produce an analogous theorem for codes. Additionally, we construct non-linear formally self-dual codes and relate them to lattices.

Key Words: Jacobi forms, unimodular lattices, self-dual codes, shadows.

1 Introduction

The shadow of a lattice has received some attention since the landmark paper [7] where it was employed to derive upper bounds on the minimum norm of unimodular odd lattices. The shadow of a code was described in [6] and numerous papers have generalized these results. In [9], a careful study of congruence properties of norms of vectors led to extension constructions for unimodular lattices and self-dual codes. Building on these latter results, in the present note we derive closed formulae for the theta series of the two translates of the even sublattice \( L_0 \) of an odd unimodular lattice \( L \), that constitute the shadow of \( L \). These formulae can be made more explicit in the case of a lattice obtained via Construction \( A_{2k} \) from a code over \( \mathbb{Z}_{2k} \). In a similar manner we derive an analogous theorem for self-dual codes over \( \mathbb{Z}_{2k} \).

An important tool is the Jacobi theta series introduced in [11] and studied further in [5].

2 Definitions and Notations

2.1 Lattices

An \( n \)-dimensional lattice is a discrete additive subgroup of \( \mathbb{R}^n \). We attach the standard inner-product, i.e. for vectors \( x \) and \( y \)

\[
x \cdot y = \sum x_i y_i.
\]
The norm of \( x \) in \( \mathbb{R}^n \) is \( x \cdot x \). The dual \( L^* \) of a lattice \( L \) is defined as

\[
L^* := \{ y \in \mathbb{R}^n | \forall x \in L, x \cdot y \in \mathbb{Z} \}.
\]

A lattice is unimodular if it is equal to its dual. A unimodular lattice is Type II if all its vectors have even norms, Type I otherwise. Consider a Type I lattice \( L \). Let \( L_0 \) denote the sublattice of even norm vectors of \( L \) and \( L_2 \) its unique nontrivial coset in \( L \). Call further \( L_1 \) and \( L_3 \) the other two cosets of \( L_0 \) in \( L_0^* \). The unique nontrivial coset of \( L \) in \( L_0^* \) is called the shadow of \( L \) (denoted by \( S \)) and is equal to \( L_1 \cup L_3 \).

### 2.2 Theta series

The ordinary theta series of a lattice \( L \) is

\[
\theta_L(\tau) := \sum_{x \in L} q^{x \cdot x},
\]

where \( q = \exp(i\pi\tau) \), with \( \tau \in \mathbb{C} \) and \( \Im(\tau) > 0 \).

The Jacobi theta series attached to a lattice \( L \) and a vector \( y \in \mathbb{R}^n \) is

\[
\theta_{L,y}(\tau, z) := \sum_{x \in L} q^{x \cdot y} \xi^{y \cdot x}
\]

where \( q \) is as before and \( \xi = \exp(2\pi iz) \), with \( z \in \mathbb{C} \). For each \( k \) and \( i = 0, 1, 2, \ldots, 2k - 1 \) put

\[
t_i(\tau, z) = \sum_{r \equiv i \pmod{2k}} q^{2\pi \xi r},
\]

where \( q \) and \( \xi \) are as before and let \( T_i(\tau) = t_i(\tau, 0) \). Further, for any real \( a \) let

\[
t_{i,a}(\tau, z) := t_i(\tau, az).
\]

### 2.3 \( \mathbb{Z}_{2k} \)-Codes

A linear code over \( \mathbb{Z}_{2k} \) is a submodule of \( \mathbb{Z}^n_{2k} \). We attach the standard inner product to the space, that is \([v, w] = \sum v_i w_i\). The dual \( C^\perp \) is understood with respect to this inner product. A code is self-dual if it is equal to its dual. The
Euclidean weight of a vector $x = (x_1, x_2, \ldots, x_n)$ is $\sum_{i=1}^n \min \{x_i^2, (2k - x_i)^2\}$.

A code is Type II if all vectors in the code have Euclidean weights which are $0 \pmod{4k}$ and Type I otherwise. If $C$ is a Type I code over $\mathbb{Z}_{2k}$ and $C_0$ is the subcode of vectors whose Euclidean weight is $0 \pmod{4k}$ then $C_2 = C - C_0$ and the shadow is $C_0^1 - C = C_1 \cup C_3$, see [1] for a complete description.

We shall recall the standard $A_{2k}$ construction of a lattice from a self-dual code over $\mathbb{Z}_{2k}$. Define the reduction modulo $2k$, by $\rho : \mathbb{Z}^n \to \mathbb{Z}_{2k}^n$, by

$$\rho(x_1, \ldots, x_n) = (x_1 \pmod{2k}, \ldots, x_n \pmod{2k}).$$

Given a code $C$ over $\mathbb{Z}_{2k}$ we construct a lattice by

$$\Lambda(C) = \frac{1}{\sqrt{2k}} \{ x \in \mathbb{Z}^n \mid \rho(x) \in C \}. \quad (1)$$

It is shown in [1] that if $C$ is a Type I code then $\Lambda(C)$ is a Type I unimodular lattice, and that if $C$ is a Type II code then $\Lambda(C)$ is a Type II unimodular lattice and that the minimum norm of the lattice is $\min \{2k, \frac{d_E}{2k}\}$, where $d_E$ is the minimum Euclidean weight of the code. Moreover, it is shown that the image of the shadow under $\Lambda$ is the shadow of the image, see [9] for a complete explanation of the connection between shadow codes and shadow lattices.

A special code we shall use later is the even code $E_n$ over $\mathbb{Z}_4$ which is defined as $E_n := 2\mathbb{Z}_4^n$. Its complete weight enumerator (defined below) is

$$cwe_{E_n} = (x_0 + x_2)^n.$$  

### 2.4 Weight Enumerators

Define the complete weight enumerator for a code $C$ over $\mathbb{Z}_{2k}$ by

$$cwe_C(x_0, x_1, \ldots, x_{2k-1}) = \sum A_{a_0, a_1, \ldots, a_{2k-1}} x_0^{a_0} x_1^{a_1} \cdots x_{2k-1}^{a_{2k-1}} \quad (2)$$

where there are $A_{a_0, a_1, \ldots, a_{2k-1}}$ vectors with $a_i$ coordinates with an $i$. The symmetric weight enumerator is

$$swe_C(x_0, x_1, \ldots, x_{2k-1}) = \sum A_{a_0, a_1, \ldots, a_k} x_0^{a_0} x_1^{a_1} \cdots x_k^{a_k} \quad (3)$$

\[4\]
where there are \( A_{a_0, a_1, \ldots, a_k} \) vectors with \( a_i \) coordinates with an \( \pm i \). The Hamming weight enumerator is given by \( H_C(x, y) = s(x, y, \ldots, y) \). The minimum Euclidean and Hamming weights of a code are denoted by \( d_E \) and \( d_H \). The Lee weight of a vector over \( \mathbb{Z}_4 \) is the sum of the Lee weights of each component. The elements have Lee weight corresponding to their binary image under the gray map, specifically, 0, 1, 2, 3 have Lee weight 0, 1, 2, and 1 respectively. The minimum Lee weight of a \( \mathbb{Z}_4 \) code is denoted \( d_{Lee} \).

We introduce the following weight enumerator. For a code \( C \) and a vector \( y \) define

\[
J_{C,y} = \sum_{c \in C} x_{i,j}^{n_{i,j}(c)}
\]

where \( n_{i,j}(c) \) is the number of coordinates that have an \( i \) in \( c \) and a \( j \) in \( y \).

Observe that for \( c \in C \),

\[
c \cdot y = \sum_{i,j} n_{i,j}(c) i j.
\]

3 Evaluations

3.1 Lattices

We shall state the main result of this section and then give the necessary lemmas to prove this theorem. The main result of this section is the following.

**Theorem 1** Let \( L \) be an odd unimodular lattice of dimension \( n \). Let \( L_0 \) denote the sublattice of even norm vectors with \( L_2 \) the unique non-trivial coset in \( L \), and let \( L_1 \) and \( L_3 \) be the other two cosets in \( L_0^* \) with the shadow \( S = L_1 \cup L_3 \). Set

\[
\mu_n(\tau) = \exp \left( \frac{i \pi n}{2}(1 - \frac{1}{\tau}) \right).
\]

Let \( y \) denote an arbitrary element of \( L_1 \). Then if \( n \equiv 0 \pmod{2} \) then the theta series \( \Theta_1 \) and \( \Theta_3 \) of \( L_1 \) and \( L_3 \) evaluate as

\[
2\Theta_1(\tau) = \left( \frac{i}{\tau} \right)^{n/2} \left( \theta_L(1 - \frac{1}{\tau}) + \mu_n(\tau) \theta_{L,y}(1 - \frac{1}{\tau}, \frac{1}{\tau}) \right)
\]

\[
2\Theta_3(\tau) = \left( \frac{i}{\tau} \right)^{n/2} \left( \theta_L(1 - \frac{1}{\tau}) - \mu_n(\tau) \theta_{L,y}(1 - \frac{1}{\tau}, \frac{1}{\tau}) \right)
\]
If \( n \equiv 1 \pmod{2} \) then

\[
\Theta_1(\tau) = \Theta_3(\tau) = \frac{1}{2} \theta_S(\tau).
\]

We prepare for the proof by a pair of lemmata. First we note the immediate.

**Lemma 1** \( \Theta_1(\tau) + \Theta_3(\tau) = \left( \frac{i}{\tau} \right) \pi \theta_L(1 - \frac{1}{\tau}) \)

**Proof:** We express \( \theta_S(\tau) \) in two ways by \( S = L_1 \cup L_3 \) and by [8, (4) p. 440], that is

\[
\theta_{L_0^*}(\tau) - \theta_L(\tau) = \left( \frac{i}{\tau} \right) \pi \theta_L(1 - \frac{1}{\tau}). \tag{5}
\]

\( \square \)

We proceed by generalizing [8, (4) p. 440] from the theta series to the Jacobi theta series. That is, we express the Jacobi theta series of the shadow as a function of the Jacobi theta series of the lattice.

**Lemma 2** For a Type I unimodular lattice \( L \) and any vector \( y \in \mathbb{R}^n \) we have

\[
\theta_{S_y}(\tau, z) = \left( \frac{i}{\tau} \right)^n \left( 2 \pi i \right)^{z^2(\tau \cdot y) / \tau} \theta_{L_y}(1 - \frac{1}{\tau}, \frac{z}{\tau}).
\]

**Proof:** First we express \( \theta_{L_0, y} \) as a function of \( \theta_{L, y} \).

\[
\theta_{L_0, y}(\tau, z) = \frac{1}{2} \left( \theta_{L, y}(\tau, z) + \theta_{L, y}(\tau + 1, z) \right).
\]

Then we use the Poisson Jacobi formula [5, 11] to express \( \theta_{L_0, y} \) as a function of \( \theta_{L_0^*, y} \) and \( \theta_{L, y} \) as a function of \( \theta_{L, y^*} \). The result follows. \( \square \)

We can now sketch a proof of Theorem 1.

**Proof:** We compute \( \Theta_1 - \Theta_3 \) by splitting the range of summation in the defining equation of \( \theta_{S_y}(\tau, 1) \) and using the tables for \( n \equiv 0 \pmod{2} \) in [9] which give the orthogonality relations between the cosets \( L_i \), to observe that the power of \( \xi \) is a constant for \( x \in L_i \) and \( y \in L_1 \). The value of \( \theta_{S_3}(\tau, 1) \) can then be obtained from Lemma 2.
Since by Lemma 1 we know $\Theta_1 + \Theta_3$ we conclude by solving a system of two equations in two unknowns, $\Theta_1$ and $\Theta_2$.

For the cases when $n \equiv 1 \pmod{2}$ we have that the glue group is the cyclic group of order 4, and that $L_1 = -L_3$. It follows that these theta series are equal. \hfill $\Box$

### 3.2 Codes

Throughout this section let $C$ be a Type I code and $C_0$ its subcode of doubly-even vectors, and $C_2 = C - C_0$ with $S = C_0^+ - C = C_1 \cup C_3$. Let $\zeta$ denote a $g$-th root of unity. The matrix $A = (a_{ij})$ is a $2k \times 2k$ matrix with

$$a_{ij} = \frac{1}{\sqrt{2k}} \zeta^{2+ij}.$$

We shall now give an analog to Theorem 1 for codes over $\mathbb{Z}_{2k}$.

**Theorem 2** Let $C$ be a Type I code of length $n$. Let $C_0$ denote the subcode of even vectors with $C_2$ the unique non-trivial coset in $C$, and let $C_1$ and $C_3$ be the other two cosets in $C_0^+$ with shadow $C_1 \cup C_3$. Let $y$ denote a constant vector of $C_1$. Then if $n \equiv 0 \pmod{2}$ then the complete weight enumerators of $C_1$ and $C_3$ evaluate as

$$2\text{cw}\,C_1(x_0, x_1, \ldots, x_{2k-1}) = \text{cw}\,C(A(x_0, x_1, \ldots, x_{2k-1})) + (-1)^{n/2k} \hat{J}_{S_0}(\zeta_{2k}^{ij} x_{ij})$$

$$2\text{cw}\,C_3(x_0, x_1, \ldots, x_{2k-1}) = \text{cw}\,C(A(x_0, x_1, \ldots, x_{2k-1})) - (-1)^{n/2k} \hat{J}_{S_0}(\zeta_{2k}^{ij} x_{ij})$$

If $n \equiv 1 \pmod{2}$ then

$$\text{swe}_{C_1}(x_0, \ldots, x_k) = \text{swe}_{C_3}(x_0, \ldots, x_k) = \frac{1}{2} \text{swe}\,S(x_0, \ldots, x_k).$$

We have the following analog to Lemma 1.

**Lemma 3** Let $C$ be a Type I code and $A$ the matrix as defined above, then

$$\text{cw}\,C(A(x_0, x_1, \ldots, x_{2k-1})) = \text{cw}\,C_1(x_0, x_1, \ldots, x_{2k-1}) + \text{cw}\,C_3(x_0, x_1, \ldots, x_{2k-1})$$
Proof: We express \( c \text{we} S(x_0, x_1, \ldots, x_{2k-1}) \) in two ways by \( S = C_1 \cup C_3 \) and by [1, Theorem 6.2, p. 1201], that is

\[
\text{we} S(x_0, x_1, \ldots, x_{2k-1}) = \text{we} C(A(x_0, x_1, \ldots, x_{2k-1})). \tag{8}
\]

Consider the polynomial \( J_{C,y} = \sum_{c \in C} x_{ij}^{n_{ij}(c)} \). We note that for \( c \in C \), \( c \cdot y = \sum_{i,j} n_{ij}(c)ij \), and that this product is constant for \( c \in C_0 \), \( y \in C_1 \) and \( c \in C_0 \), \( y \in C_1 \). Hence, it is most useful when \( y \in S \), the shadow of the code.

From [1] (corrected in [4]) we have

\[
J_{S,y}(X_{ij}) = \frac{1}{|C|} (T \otimes I) \cdot J_{C,y}(X_{\phi(a)}) \tag{9}
\]

where \( T_{a,b} = (\zeta_{4k}^a)^b \) with \( a, b \in \mathbb{Z}_{2k} \) and \( \phi(a) = \zeta_{4k}^a(a, b) \) with \( a = (a, b) \).

Let \( y \in S \) and substitute \( X_{ij} = z_{ij} x_{i,j} \) in \( J_{S,y}(X_{ij}) \). Splitting the range of summation we have

\[
J_{S,y}(z_{ij} x_{i,j}) = z^{c \cdot y} \text{we} C_1(x_{i,j}) + z^{c \cdot y} \text{we} C_3(x_{i,j}) \tag{10}
\]

where \( c \cdot y \) represents the constant inner product of \( y \) with an element of \( C_i \). Note that it was imperative that \( y \) be a constant vector for equation 10 to hold.

Using the tables in [9] which give the orthogonality relations between the cosets \( C_i \), we get the following lemma.

Lemma 4 Let \( C \) be a Type I code then,

\[
\text{we} C_1(x_{i,j}) - \text{we} C_3(x_{i,j}) = (-1)^{\frac{1}{2}} J_{S,y}(z_{ij}^{\frac{1}{2k}} x_{i,j}) \tag{11}
\]

The proof of Theorem 2 follows directly from the previous lemmata and that fact that when \( n \) is odd, \( C_1 = -C_3 \).

We give an elementary example of Theorem 2. Let \( C = E_2 = \{(00), (22), (20), (02)\} \). Then \( C_0 = \{(00), (22)\} \), \( C_2 = \{(02), (20)\} \), \( C_1 = \{(11), (33)\} \), and \( C_3 = \{(13), (31)\} \). Then \( W_{C_1} = x_1^2 + x_3^2 \) and \( W_{C_3} = 2x_1x_3 \). Choose \( y = (11) \). We
have $J_{S,y}(X_{ij}) = x_{1}^2 + x_{13}^2 + 2x_{11}x_{13}$, then $J_{S,y}(\sqrt{-1}x_{ij}) = -x_{1}^2 - x_{3}^2 + 2x_{1}x_{3}$.

Finally,

$$W_{C}(A(x_0, x_1, x_2, x_3) = J_{S,y}(\sqrt{-1}x_{ij})$$
$$= x_{1}^2 + x_{3}^2 + 2x_{1}x_{3} + x_{1}^2 + x_{3}^2 - 2x_{1}x_{3}$$
$$= 2x_{1}^2 + 2x_{3}^2 = 2W_{C_i}$$

and

$$W_{C}(A(x_0, x_1, x_2, x_3) = J_{S,y}(\sqrt{-1}x_{ij})$$
$$= x_{1}^2 + x_{3}^2 + 2x_{1}x_{3} - x_{1}^2 - x_{3}^2 + 2x_{1}x_{3}$$
$$= 4x_{1}x_{3} = 2W_{C_3}.$$  

Note that if the vector (13) is used then the theorem does not hold since it is not a constant vector.

4 Applications

4.1 Construction $A_{2k}$

Theorem 1 can only be useful if we know how to compute $\theta_{S,y}$. Following [8] we shall denote by $[a]$ the vector

$$[a] = (a/2, \ldots, a/2).$$

We shall require the following result from [5].

Lemma 5 (Choie-Kim [5]) If $L$ is a Type I lattice obtained by Construction $A_{2k}$ from a code $C$ then

$$\theta_{L, [a]}(\tau, z) = c\epsilon_{C}(t_{0,a}(\tau, z), t_{1,a}(\tau, z), t_{2,a}(\tau, z), \ldots, t_{2k-1,a}(\tau, z)).$$

Combining this lemma with Theorem 1 we obtain

Theorem 3 With the notations of Theorem 1 we have for a Type I lattice, whose shadow contains $[a]$, the following identities hold:

$$2\Theta_{1}(\tau) = (\frac{1}{\tau})^{n/2}(c\epsilon_{C}(T_{0}(1 - \frac{1}{\tau}), T_{1}(1 - \frac{1}{\tau}), T_{2}(1 - \frac{1}{\tau}), \ldots, T_{2k-1}(1 - \frac{1}{\tau})))$$

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$$+ \mu_n \, \text{cwe}_C(t_{0,a}(1 - \frac{1}{\tau}, \frac{1}{\tau}), t_{1,a}(1 - \frac{1}{\tau}, \frac{1}{\tau}), \ldots, t_{2k-1,a}(1 - \frac{1}{\tau}, \frac{1}{\tau})))$$

$$\Theta_3(\tau) = (\text{cwe}_C(T_0(1 - \frac{1}{\tau}), T_1(1 - \frac{1}{\tau}), T_2(1 - \frac{1}{\tau}), \ldots, T_{2k-1}(1 - \frac{1}{\tau}))$$

$$- \mu_n \, \text{cwe}_C(t_{0,a}(1 - \frac{1}{\tau}, \frac{1}{\tau}), t_{1,a}(1 - \frac{1}{\tau}, \frac{1}{\tau}), \ldots, t_{2k-1,a}(1 - \frac{1}{\tau}, \frac{1}{\tau}))).$$

### 4.2 Shadow sums and extensions

The following construction while implicit in [8] was first defined in [10]. It generalizes the extensions of [9].

**Theorem 4 (Dougherty-Solé [10])** Let $L$ and $L'$ denote two Type I unimodular lattices of respective dimensions $n$ and $n'$. The set

$$L \oplus_S L' := \bigcup_{i=0}^{3} L_i \times L'_i$$

is a unimodular lattice of dimension $n + n'$. It is Type II if $n + n'$ is a multiple of 8. Let $C$ and $C'$ denote two Type I self-dual codes over $\mathbb{Z}_{2k}$ of respective lengths $n$ and $n'$. The set

$$C \oplus_S C' := \bigcup_{i=0}^{3} C_i \times C'_i$$

is a self-dual code of length $n + n'$. It is Type II if $n + n'$ is a multiple of 8.

For instance:

- $\mathbb{Z}^i \oplus_S \mathbb{Z}^{8-i} = E_8$ for $0 < i < 8$
- $D_{12}^+ \oplus_S D_{12}^+ = \text{Niemeier lattice of root system } D_{12}^2$
- $O_{23} \oplus_S \mathbb{Z} = A_{24}$ the Leech lattice.
These results give added importance to Theorems 1 and 2, since the theta series of such a lattice is easy to compute if one knows the theta series of the four cosets of $L_0$ into $L_0^*$. Specifically, if $L$ and $L'$ denote two Type I unimodular lattices of respective dimensions $n$ and $n'$, then the theta series of their shadow sum is

$$\theta_{L \oplus L'} = \sum_{i=0}^{3} \theta_{L_i} \theta_{L'_i}. $$

Additionally, if $C$ and $C'$ denote two Type I self-dual codes of respective lengths $n$ and $n'$, then the cwe of their shadow sum is

$$cwe_{C \oplus C'} = \sum_{i=0}^{3} cwe_{C_i} cwe_{C'_i}. $$

**5 Constant Vectors and Shadows**

In light of Theorem 1 we would like to know when a constant vector is contained in the shadow of a unimodular lattice. As an example we note that $[1] = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ is not in the shadow of any unimodular lattice formed by construction $A_8$ from a self-dual code over $\mathbb{Z}_8$. Since if $[1]$ were in the shadow of the lattice then there would exist a vector $s$ in the shadow of the code such that $\Lambda(s) = [1]$, where $\Lambda$ indicates the $A_8$ construction. Then for some integer $\alpha$ we have $\frac{1}{\sqrt{2}} \alpha = \frac{1}{2}$ which implies that $\sqrt{2}$ is an integer, giving a contradiction.

In general we want to know when there is a constant vector in the shadow of a code over $\mathbb{Z}_{2k}$. We shall develop a general theory and apply it to this situation.

Let $C$ be a self-dual code over $\mathbb{Z}_{2k}$. We shall give an alternate definition of a shadow and call it the generalized shadow.

Let $s$ be any vector in $\mathbb{Z}_{2k}^n$ such that $s \in S$, $s \notin C$, and $2s \in C$. Define a subcode of $C$ by

$$sC_0 = \{v \mid v \in C, \ [v, s] = 0\}$$

The code $sC_0$ is a subcode of index 2 in $C$ and let $sC_2 = C - sC_0$. Then $sC_0^\perp = C \cup sS = C \cup sC_1 \cup sC_3$.

Notice that if $L = \Lambda(C)$ is the lattice formed from $C$ then $\Lambda(sC_0) = \Lambda(s)L_0$ and $\Lambda(sS) = \Lambda(s)L_1 \cup \Lambda(s)L_3$. Specifically the $s$-shadow is mapped
via the construction to the corresponding $\Lambda(s)$ shadow of the lattice, i.e.
$sL_0 = \{ v \mid v \cdot \Lambda(s) \in \mathbb{Z}, \ v \in \mathbb{L} \}$, $sL_2 = sL - sL_0$, and $sS = sL_0^\perp - sL$.

If the vector $s \in S$ where $S$ is the standard shadow then $sC_0 = C_0$ and
$sS = S$.

Let $\eta$ be a $4k$-th root of unity, i.e. $\eta = e^{2\pi i/4k}$. First we compute the
complete weight enumerator of the standard subcode $C_0$.

$$ce_{C_0}(x_0, x_1, \ldots, x_{2k-1}) = \frac{1}{2}(ce_C(x_0, x_1, \ldots, x_{2k-1})$$

$$+ ce_C(x_0, \eta^{12} x_1, \ldots, \eta^{(2k-1)^2} x_{2k-1})).$$

Specifically the second summand replaces $x_i$ with $\eta^{i^2} x_i$.

Let $s$ be the constant vector $s = (\alpha, \alpha, \ldots, \alpha)$. Let $\mu = e^{2\pi i/4k}$. Now we can compute $ce_{sC_0}$ for this vector $s$.

$$ce_{C_0}(x_0, x_1, \ldots, x_{2k-1}) = \frac{1}{2}(ce_C(x_0, x_1, \ldots, x_{2k-1})$$

$$+ ce_C(x_0, \mu^{1\alpha} x_1, \ldots, \mu^{(2k-1)^\alpha} x_{2k-1})).$$

Specifically the second summand replaces $x_i$ with $\mu^{i\alpha} x_i$.

Moreover, note that for a given monomial $x_0^{a_0} x_1^{a_1} \cdots x_{2k-1}^{a_{2k-1}}$ representing a vector $v$ we have $[v, s] = 0$ if and only if

$$x_0^{a_0} x_1^{a_1} \cdots x_{2k-1}^{a_{2k-1}} = x_0^{a_0} (\mu^{\alpha} x_1)^{a_1} \cdots (\mu^{(2k-1)^\alpha} x_{2k-1})^{a_{2k-1}}.$$  

Hence, if this is a weight enumerator for a subcode $D_0$ then $D_0 = sC_0$.

If $S$ contains some constant vector $s = (\alpha, \alpha, \ldots, \alpha)$ then $ce_{C_0} = ce_{sC_0}$ and therefore

$$ce_C(x_0, \eta^{12} x_1, \ldots, \eta^{(2k-1)^2} x_{2k-1}) = ce_C(x_0, \mu^{1\alpha} x_1, \ldots, \mu^{(2k-1)^\alpha} x_{2k-1}).$$

(13)

**Theorem 5** A shadow of a self-dual code $C$ over $\mathbb{Z}_{2k}$ has a constant vector
in the shadow $S$ if and only if equation (13) holds for some $\alpha$.

**Example:** Let $C$ be the self-dual code in $\mathbb{Z}_4^2$, $C = \{00, 02, 20, 22\}$. With
respect to the above $k = 1$ and in equation 13 we have $\eta = exp \frac{2\pi i}{8}$ and $\mu = i$.  

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Then \( \text{cw} \ e \ C(x_0, x_1, x_2, x_3) = x_0^2 + 2x_0x_2 + x_2^2 \), and
\[
\text{cw} \ e \ C(x_0, \eta^{12} x_1, \ldots, \eta^{(2k-1)^2} x_{2k-1}) = x_0^2 - x_0x_2 + x_2^2
= \text{cw} \ e \ C(x_0, \mu^1, \ldots, \mu^{(2k-1)^2} x_{2k-1}).
\]

Hence we see that the shadow contains the all-one vector.

Let \( s = (\alpha, \alpha, \ldots, \alpha) \), we can compute \( \text{cw} \ e \ s \ S(x_0, \ldots, x_{2k-1}) \) easily since \( s \ S = (s + C) \), hence if \( v \in C, v = (v_1, \ldots, v_n) \) then \( s + v = (\alpha + v_1, \ldots, \alpha + v_n) \).

This gives
\[
\text{cw} \ e \ s \ S(x_0, \ldots, x_{2k-1}) = \text{cw} \ e \ C(x_\alpha, x_{1+\alpha}, \ldots, x_{2k-1+\alpha}) \quad (14)
\]

Moreover, given that
\[
\text{cw} \ e \ s \ C_2(x_0, \ldots, x_{2k-1}) = \text{cw} \ e \ C(x_0, \ldots, x_{2k-1}) - \text{cw} \ e \ s \ C_0(x_0, \ldots, x_{2k-1}),
\]
we have
\[
\text{cw} \ e \ s \ C_1(x_0, \ldots, x_{2k-1}) = \text{cw} \ e \ s \ C_0(x_0, x_{1+\alpha}, \ldots, x_{2k-1+\alpha}) \quad (15)
\]
and
\[
\text{cw} \ e \ s \ C_3(x_0, \ldots, x_{2k-1}) = \text{cw} \ e \ s \ C_2(x_0, x_{1+\alpha}, \ldots, x_{2k-1+\alpha}) \quad (16)
\]

So if the complete weight enumerator of \( C \) is known then it is easy to compute the complete weight enumerators of \( \text{cw} \ e \ s \ C_0, \text{cw} \ e \ s \ C_2, \text{cw} \ e \ s \ C_1, \text{cw} \ e \ s \ C_3, \) and \( \text{cw} \ e \ s \ S \). Moreover, the theta series of the corresponding lattices can also be computed.

Given \( s = (\alpha, \alpha, \ldots, \alpha) \), a corresponding vector in the induced lattice is \( \frac{1}{\sqrt{2k}}(\alpha, \alpha, \ldots, \alpha) \) is in the s-shadow of the lattice. Hence it will be interesting to know when there exists a constant vector \( S \) such that \( s + s \in C \) for a self-dual code \( C \) over \( \mathbb{Z}_{2k} \).

**Theorem 6** Let \( C \) be a self-dual code over \( \mathbb{Z}_{2k} \) then \((k, k, \ldots, k) \in C. \)

**Proof:** If \( x \in \mathbb{Z}_{2k} \) then \( xk = 0 \) if \( x \equiv 0 \pmod{2} \) and \( xk = k \) if \( x \equiv 1 \pmod{2} \).

Let \( v \in C \), we have \([v, v] = 0 \). If \( v_i \equiv 0 \pmod{2} \) then \( v_i^2 \equiv 0 \pmod{2} \) and if \( v_i \equiv 1 \pmod{2} \) then \( v_i^2 \equiv 1 \pmod{2} \). Hence there are evenly many \( i \) (denote the number by \( 2r \)) such that \( v_i \equiv 1 \pmod{2} \). Therefore
\[
[v, (k, k, \ldots, k)] = 2rk = 0.
\]

\[\square\]
Corollary 1 A unimodular lattice constructed from some code via construction $A_{2k}$ contains the constant vector $\frac{1}{\sqrt{2k}}(k, k, \ldots, k)$.

An important example of the previous corollary is that any lattice constructed from a self-dual code over $\mathbb{Z}_4$ contains the all-one vector.

Theorem 7 If $C$ is a self-dual code over $\mathbb{Z}_{2^r}$ of length $n \neq 0 \pmod{2^r}$ then there exists a constant vector $s$, such that $s \notin C$ but $s + s \in C$.

Proof: Theorem 6 gives that $(2^{r-1}, 2^{r-1}, \ldots, 2^{r-1}) \in C$. There exists $\alpha$ such that

$$(2^\alpha, 2^\alpha, \ldots, 2^\alpha) \notin C$$

and

$$(2^{\alpha+1}, 2^{\alpha+1}, \ldots, 2^{\alpha+1}) \in C.$$ 

Otherwise we would have $(1, 1, \ldots, 1) \in C$, but

$$[(1, 1, \ldots, 1), (1, 1, \ldots, 1)] = n \neq 0 \pmod{2^r}.$$ 

Hence $s = (2^\alpha, 2^\alpha, \ldots, 2^\alpha)$. \hfill \square

If $E_n := 2\mathbb{Z}_4^n$ then $cwE_n = (x_0 + x_2)^n$. Computing the left hand of Equation 13 we have $(x_0 - x_2)^n$ and computing the right side for $\alpha = 1$ we have $(x_0 - x_2)^n$. So the all one vector is in the shadow and is not in the code, i.e. $S = sS$, where $s = (1, 1, \ldots, 1)$. Then the associated lattice is in the desired situation for Theorem 1.

Over $\mathbb{Z}_{k^2}$ with $k$ even we have the natural generalization of the $E_n$ given where $(k)$ generates a self-dual code of length 1 over $\mathbb{Z}_{k^2}$.

If $C_n = (k) \times (k) \times \cdots \times (k)$ then $(k, k, \ldots, k) \in C_n$ but $(\frac{k}{2}, \frac{k}{2}, \ldots, \frac{k}{2}) \notin C_n$.

The complete weight enumerator is easily determined, i.e.

$cwE_n(x_0, \ldots, x_{k^2-1}) = (x_0 + x_k)^n$.

The left hand side of Equation 13 gives $(x_0 - x_k)^n$ since $\eta_{2k}^{k^2} = -1$ and the right hand side of Equation 13 gives $(x_0 - x_k)^n$ since $\mu^{i(\frac{k}{2})} = (e^{2\pi i k})^{\frac{k}{2}} = -1$.

In general, the lattice formed under the image of this code contains the vector

$$\Lambda_{k^2}(\frac{k}{2}, \frac{k}{2}, \ldots, \frac{k}{2}) = \frac{1}{\sqrt{k^2}}(\frac{k}{2}, \frac{k}{2}, \ldots, \frac{k}{2}) = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) = [1]$$.
6 Formally Self-Dual Codes

A code $C$ is said to be formally self-dual with respect to a weight enumerator if the weight enumerator is held invariant by the MacWilliams relations.

**Theorem 8** Let $C$ be a Type I code over $\mathbb{Z}_{2k}$ with odd length $n$. The codes $D_1 = C_0 \cup C_1$ and $D_3 = C_0 \cup C_3$ are formally-self dual (with respect to the symmetric or Hamming weight enumerators) non-linear codes.

**Proof:** Let $W_C(X)$ denote either the symmetric or Hamming weight enumerator. We note that $W_{C_1}(X) = W_{C_3}(X) = \frac{1}{2}W_S(X)$ since $n$ is odd. Let $M \cdot W_C(X)$ denote the action of the variable transformation given by the MacWilliams relations. Apply the MacWilliams relations to $W_{D_1}(X)$ and the result is:

$$
\frac{1}{|D_1|} (M \cdot W_{D_1}(X)) = \frac{1}{|C|} (M \cdot W_{C_0}(X) + M \cdot (\frac{1}{2})(W_{C_0^+(X)} - W_C(X)))
$$

$$
= \frac{1}{2}W_{C_0^+(X)} + \frac{|C_0^+|}{2|C|} W_{C_0}(X) - \frac{|C|}{2|C|} W_C(X)
$$

$$
= \frac{1}{2}W_{C_0^+(X)} + W_{C_0}(X) - \frac{1}{2}W_C(X)
$$

$$
= \frac{1}{2}W_C(X) + \frac{1}{2}W_S(X) + W_{C_0}(X) - \frac{1}{2}W_{C_0}(X) - \frac{1}{2}W_{C_2}(X)
$$

$$
= \frac{1}{2}W_{C_0}(X) + \frac{1}{2}W_C(X) + \frac{1}{2}W_S(X) + \frac{1}{2}W_{C_0}(X) - \frac{1}{2}W_{C_2}(X)
$$

$$
= W_{C_0}(X) + \frac{1}{2}W_S(X)
$$

$$
= W_{D_1}(X).
$$

The same computation holds for $D_3$, since $W_{D_1}(X) = W_{D_3}(X)$. The code is nonlinear since the glue group is the cyclic group of order 4.

As a simple example we take the self-dual code of length 1. Then $D_1 = \{0, 1\}$, and $sw_{D_1} = x_0x_1$. Note that applying the MacWilliams relations results in $x_0^1x_1^1$, but that the same is not true for the complete weight enumerator.

Let the minimum weight of $C_i$ be denoted by $d_i$ then this theorem is especially useful when $d_2 < d_i$ for $i = 0, 1, 3$. Then a code is produced with higher minimum weight than the self-dual code with a weight enumerator that satisfies the MacWilliams relations.
Corollary 2  Let $C$ be a Type I code of odd length, with $D_1$ and $D_3$ as defined above, then $A_{2k}(D_1)$ and $A_{2k}(D_3)$ are sphere packings whose theta series are held invariant by the Poisson formula, that is

$$\Theta_L(z) = (\det L)^{\frac{1}{2}}(\frac{i}{z})^\frac{L}{2}\Theta_L\left(\frac{-1}{z}\right),$$

and whose minimum norm is $\min\{2k, d_E(D_i)\}$ where $d_E(D_i)$ is the minimum Euclidean weight of $D_i$, for $i = 1, 2$.

We computed the sve of fsd codes obtained from cyclic self-dual $Z_4$ codes of [12]. Some have a better minimum weight than the self-dual codes of the same length ([13], Table XVI, P. 279). This is the case for lengths 7, 15, 23 and 47. Based on the following data and polarization computations akin to [3], we conjecture that the codewords of fixed Lee composition support $t$-designs with

- $t = 2$ for lengths 7, 15, 21, 31, 47
- $t = 3$ for length 23.

Borrowing the notations of [12], we give the parameters of our formally self-dual codes in lengths 7, 15, 21, 23, 31, 35 and 47. Until length 23, we use a “*” when the parameter is better than any one known for this length.

- **Length 7**
  - From the only non trivial cyclic self-dual code $C(7, 4^32, 4)$, we construct a formally self-dual code with $d_H = 4*$, $d_{Lee} = 5*$, $d_E = 7*$ and
    
    $$sve := a^7 + 7a^3c^4 + 42a^2b^4c + 14c^3b^4 + 28a^3cb^3 + 28ac^3b^3 + 8b^7.$$  

- **Length 15**
  - From the only non trivial cyclic self-dual code $C(15, 4^42^7, 6)$, we construct a formally self-dual code with $d_H = 4*$, $d_{Lee} = 7*$, $d_E = 7$ and
    
    $$sve := 105a^{11}c^4 + 280a^9c^6 + 435a^7c^8 + 168a^5c^{10} + 35a^3c^{12} + 3360a^5b^7a^2 + a^{15} + 5040b^8a^5c^2 + 8400b^8a^3c^4 + 1680b^8ac$$
    $$+ 3360a^8b^7c^2 + 8400a^4b^7c^4 + 120a^8b^7 + 120c^8b^7 + 1024b^{15} + 240b^8a^7.$$  

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• Length 21
There are four inequivalent non trivial cyclic self-dual codes: $C_1 := C_{21,1}(21,4^{6^2}2^9, 6)$, $C_2 := C_{21,2}(21,4^{3^2}2^5, 4)$, $C_3 := C_{21,3}(21,4^{9^2}, 4)$ and another one $C_4$ generated by $(fh,2fg)$ with $f := f_1f_2^*$, $h := z^3 - 1$ and $fgh = z^{21} - 1$ with the notation of [12]. We obtain:

<table>
<thead>
<tr>
<th>Code</th>
<th>$d_E$</th>
<th>$d_{L_{ee}}$</th>
<th>$d_H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>8</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>$C_2$</td>
<td>8</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>$C_3$</td>
<td>8</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>$C_4$</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

• Length 23
From the only non trivial cyclic self-dual code $C(23,4^{11}2,10)$, we construct a formally self-dual code with $d_H = 8$, $d_{L_{ee}} = 11$, $d_E = 15$ and $swc := a^{12} + 8096b_1^6a_7 + 506a_1^5c_8 + 1288a_1^{11}c$.

12 + 253$ a^7 c^{10} + 127512 a_1^{10} b^7 c^6 + 2024 e^{14} b^7$

$2 + 8096 b_1^5 a_8 + 2576 b_1^{12} c_1 + 8096 b_1^{15} e^8 +$

$202400 a^8 b^7 c^8 + 226688 b_1^5 a_9^6 c^2 + 28336 a^4 c_1^{12} b^7 + 1020096 b_1^{11} a_7 c^5 + 170016 b_1^{16} a_5 c^2 + 56672 a_1^4 c_4 + 15456 b_1^{11} a_1^6 c + 1020096 b_1^{11} a_5^6 c^2 + 15456 b_1^{11} c_1^6 a + 56672 b_1^{16} a_6 c^2 + 12$

$7512 c_1^{10} b_7 c^6 + 283360 b_1^{16} a_3 c^4 + 28336 b_1^{12} a_1^{10} c + 226688 b_1^{15} c^6 a^2 + 2024 a_1^{4} b^7 c^2 + 42$

$5040 b_1^{12} a_8 c + 85008 b_1^{12} a_4 c^7 + 1190112 b_1^{12} a_6 c^3 + 318780 b_8 a_9^6 c^6 + 85008 b_8 a_1^{11} c$

$+ 1204 b_8 a_5 c^2 + 141680 b_1^{12} a_9^6 c^2 + 28336 a_1^{12} c_4 b^7 + 404800 b_8 a_7 c^8 + 28336 b_8^5 a^3 c^2 + 191268 b_8^5 a^5 c^4 + 283360 b_1^{16} c^9 a^3 + 283360 b_1^{16} a_9^3 c + 2048 b_2^{23} + 1012 b_8 a_4 c^2.$

• Length 31
There are five inequivalent non trivial cyclic self-dual codes: $C_1 := C_{31,1}(31,4^{5^2}2^{21},6)$, $C_2 := C_{31,2}(31,4^{10}2^{11},10)$, $C_3 := C_{31,3}(31,4^{10}2^{11},10)$, $C_4 := C_{31,4}(31,4^{15}2,12)$ and $C_5 := C_{31,5}(31,4^{15}2,12)$ with the notation of [12]. The codes $C_2$ and $C_3$ have the same symmetric weight enumerator as do $C_4$ and $C_5$. We obtain:

<table>
<thead>
<tr>
<th>Code</th>
<th>$d_E$</th>
<th>$d_{L_{ee}}$</th>
<th>$d_H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>15</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>$C_2, C_3$</td>
<td>15</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>$C_4, C_5$</td>
<td>15</td>
<td>13</td>
<td>8</td>
</tr>
</tbody>
</table>
Length 35, there exist four inequivalent cyclic self-dual codes. We have, borrowing the notations of [12]:

<table>
<thead>
<tr>
<th>codes</th>
<th>generators</th>
<th>$d_{L_{ee}}$</th>
<th>$d_E$</th>
<th>$d_H$</th>
<th>$t$-design</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$f_3 f_1 h_0, 2 f_3 f_1 f_3 f_1^*$</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>$t = 1$</td>
</tr>
<tr>
<td>2</td>
<td>$f_3^2 f_1 f_0, 2 f_3 f_1 f_3 f_1^*$</td>
<td>8</td>
<td>8</td>
<td>6</td>
<td>$t = 1$</td>
</tr>
<tr>
<td>3</td>
<td>$f_3^2 f_3 h_0, 2 f_3 f_1 f_3 f_1^*$</td>
<td>6</td>
<td>8</td>
<td>3</td>
<td>$t = 1$</td>
</tr>
<tr>
<td>4</td>
<td>$f_3 f_1 f_1^2 h_0, 2 f_3 f_3^*$</td>
<td>4</td>
<td>8</td>
<td>2</td>
<td>$t = 1$</td>
</tr>
</tbody>
</table>

and we obtain four formally self-dual codes with minimum weights respectively $d_{L_{ee}} = 6$, $d_E = 8$, $d_H = 4$ for the first code, $d_{L_{ee}} = 8$, $d_E = 8$, $d_H = 6$ for the second code, $d_{L_{ee}} = 8$, $d_E = 8$, $d_H = 4$ for the third code and $d_{L_{ee}} = 4$, $d_E = 8$, $d_H = 2$ for the fourth code. Their symmetric weight enumerators can be polarized at most one time. This indicates that these codes cannot contain $t$-design with $t > 1$.

Length 39
There is a unique non-trivial self-dual cyclic code $((fh, 2ff^*)$ in the notation of [12]). From this code, we construct a formally self-dual code. The symmetric weight enumerators of the two codes can be polarized at most one time. Their parameters are:

<table>
<thead>
<tr>
<th>cyclic code</th>
<th>FSD code</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_H = 3$</td>
<td>$d_H = 6$</td>
</tr>
<tr>
<td>$d_{L_{ee}} = 6$</td>
<td>$d_{L_{ee}} = 12$</td>
</tr>
<tr>
<td>$d_E = 12$</td>
<td>$d_E = 15$</td>
</tr>
</tbody>
</table>

Length 47
We construct a formally self-dual code from the quadratic residue code over $Z_7$ with minimum weight respectively $d_{L_{ee}} = 17$, $d_E = 23$, $d_H = 12$ and $\psi = 356730 a^{31} c^{16} + 2330636 a^{27} c^{20} + 12972 c^{38} c^{12} + 4$

\[324 c^{36} a^{11} + 3840840 a^{23} c^{24} + 1664740 c^{28} a^{19} + 178365\]

\[+ c^{32} a^{15} + a^{47} + 1061836032 a^{22} b^{23} c^{2} + 745803520 a\]

\[1947 c + 5876246816 c^{21} a^{15} b^{11} + 634538352 c^{25} a^{11} b^{11} + 7387648 b^{24} a^{23} + 311328 b^{30} c^{27} + 53271680 b^{28} c\]

\[19 + 91322880 b^{32} a^{15} + 35422208 b^{36} c^{11} + 1163320312 b^{12} c^{25} a^{10} + 28743591096 b^{12} a^{18} c^{17} + 25717949928 b^{12} a\]
16c^{19}+10654336b^{12}c^{29}a^6+259440b^{12}c^{31}a^4+44139392b^{12}a^{28}c^7+14690617040b^{12}a^{14}c^{21}+9354
057312b^{12}a^{22}c^{13}+20566863856b^{12}a^{20}c^{15}+444654216b
12a^{26}c^9+95128b^{12}a^{32}c^3+5287075872b^{12}c^{23}a^{12}+1608528b^{12}a^{30}c^5+2643909800b^{12}a^{24}c^{11}+
8648b^{12}c^{33}a^2+148218072b^{12}c^{27}a^8+1883169108480
b^{24}a^7c^{16}+176972672b^{24}a^7c^{22}+
10386094688256b^{24}a^{11}
c^{12}+6277489109760b^{24}a^9c^{14}+258497022080b^{24}a^5c^{18}+8788484753664b^{24}a^{13}c^{10}+
68005104640b^{24}a^1
c^8+1946699392b^{24}a^{21}c^2+13601020928b^{24}a^3c
20\cdot 7335233600b^{16}c^{24}a^7+393286739120b^{16}a^{19}c^{12}+90198640b^{16}a^{27}c^4+
22005700800b^{16}a^{23}c^8+123
589156064b^{16}a^{21}c^{10}+471246816b^{16}c^{26}a^5+
2042069536b
16a^{25}c^6+837531264768b^{16}a^{15}c^{16}+1037760b^{16}a
29c^2+
12885520b^{16}c^{28}a^3+235972043472b^{16}c^{20}a
11\cdot 56176889120b^{16}c^{22}a^9+
739098898560b^{16}a^{17}c^{14}+
574854698880b^{16}c^{18}a^{13}+69184b^{16}c^{30}a+8405856b^{20}a^{26}c+101596704b^{20}c^{25}a^2+
506446559296b^{20}
a^{12}c^{15}+86214886912b^{20}c^{21}a^6+646822836000b^{20}c^{19}a
8\cdot 258644660736b^{20}a^{30}c^7+1365514876000b^{20}a^{18}c^9
+846639200b^{30}a^{24}c^3+5843614106880b^{30}a^{14}c^{13}+5118232320b^{20}c^{23}a^4+23543868672b^{20}a^{22}c^5+
2457673355808b^{20}a^{10}c^{17}+3798222458976b^{20}a^{16}c
11\cdot 4026380117760b^{28}a^8c^{11}+1012161920b^{28}a^{18}c+4921131255040b^{28}a^{10}c^9+
9109457280b^{28}c^{17}a^2+206481031680b^{28}c^{13}a^4+2684253411840b^{28}a^{12}c^7+
619443095040b^{28}a^{14}c^5+1445367221760b^{28}c^{13}a^6
19
\[ + 51620257920 b^{28} a^{16} c^3 + 274242608640 b^{32} a^5 c^{10} + 41551910400 b^{32} a^3 c^{12} + 124655731200 b^{32} a^{11} c^4 + 1369843200 b^{32} a c^{14} + 457071014400 b^{32} a^9 c^6 + 587662732 \]
\[ 800 b^{32} a^7 c^8 + 9588902400 b^{72} a^{13} c^2 + 16365060096 b \]
\[ 36 c^5 a^6 + 389644288 b^{36} a^{10} c + 5844664320 b^{36} a^8 c^3 + 11689328640 b^{36} c^7 a^4 + 1948221440 b^{36} c^9 a^2 + 166 \]
\[ 207641600 a^4 b^{31} c^{12} + 9076923504 a^{19} c^{17} b^{11} + 9076923504 a \]
\[ 17 c^{19} b^{11} + 3113280 a^{27} b^{19} c + 98812048 c^{27} b^{11} a^9 \]
\[ + 311328 c^{31} a^{5} b^{11} + 2440188864 a^{13} c^{23} b^{11} + 42081583104 c^5 a^{7} b^{35} + 9132288 c^{29} a^{7} b^{11} + 17296 \]
\[ c^{23} b^{11} a^3 + 2824753662720 a^8 b^{23} c^{16} + 637599744 b \]
\[ 35 c^{11} a + 637599744 b^{35} a^{11} c + 17296 a^{33} c^3 b^{11} + 2440188864 a \]
\[ 23 c^{13} b^{11} + 3693824 a^{24} b^{23} + 51699404160 a^{18} b^{23} c^6 + 44941511296 c^{10} a^{22} b^{15} + 40803062784 a^{30} b^{23} c^4 \]
\[ + 25771040 \ a^{28} b^{15} c^4 + 7532986931712 a^{10} b^{23} c^{14} + 98812048 c^9 a^{27} b^{11} + 328488399360 c^{18} a^{14} b^{15} + 223 \]
\[ 4248505280 c^{11} a^{17} b^{19} + 44941511296 a^{10} c^{22} b^{15} + 157314695648 c^{20} a^{12} b^{15} + 276736 c^{30} a^{2} b^{15} + 425 \]
\[ 10800640 \ a^{3} c^{17} b^{27} + 9132288 c^7 a^{29} b^{11} + 7335233600 c^{8} \]
\[ a^{24} b^{15} + 2824753662720 c^{8} a^{16} b^{23} + 3895742737920 c \]
\[ 15 a^{13} b^{19} + 157314695648 a^{20} c^{12} b^{15} + 634538352 a^{25} \]
\[ c^{11} b^{11} + 7532986931712 c^{10} a^{14} b^{23} + 7335233600 c^{24} a^8 \]
\[ b^{15} + 11689328640 a^9 c^3 b^{35} + 10386094688256 c^{12} a^{12} b^{23} + 42081583104 b^{35} c^7 a^5 + 11689328640 b^{35} c^9 a^3 \]
\[ + 25771040 \ a^{28} a^{4} b^{15} + 628329088 c^{36} a^6 b^{15} + 123164124160 c^{21} a^7 b^{19} + 71869204000 c^{19} a^9 b^{19} + 328488399360 \]
\[ a^{18} b^{15} c^{14} + 10236464640 c^{25} a^5 b^{19} + 51699404160 c^{18} \]
\[ b^{23} a^6 + 5876246816 a^{21} c^{15} b^{11} + 311328 a^{31} c^5 \]
\[ b^{11} + 628329088 a^{30} b^{15} + 3895742737920 c^{13} a^{15} b^{19} + 40803062784 c^{20} b^{23} a^4 + 418765632384 a^{16} c^{16} b^{15} + 3113280 \ c^{27} b^{19} a + 1061386032 e^{22} b^{23} a^2 + 338655680 e^{25} b^{19} a^3 + 3693824 e^{24} b^{23} + 2234248505280 a^{11} b^{19} c^{17} + 123164124160 a^{21} b^{19} c^7 + 718692040000 a^{19} c^9 b^{19} + 8388608 b^{47} + 10236464640 a^{23} b^{19} c^5 + 338655680 a^{25} b^{19} c^3 + 276736 a^{30} c^{21} b^{15} + 2890734443520 a^7 c^{13} b^2 7 + 2890734443520 c^7 b^{27} a^{13} + 745803520 e^{19} b^{27} a + 4251080 0640 a^{17} b^{27} c^3 + 578146888704 a^{5} b^{27} c^{15} + 6263257960960 , c^{9} b^{27} a^{11} + 6263257960960 e^{11} b^{27} a^9 + 578146888704 a^{1} 5b^{27} c^5 + 731313623040 a^6 e^{10} b^{31} + 10958745600 a^2 e^{14} b^{31} c^{31} + 166207641600 e^4 b^{31} a^{12} + 91322880 a^{16} b^{31} + 91322880 c^{16} b^{31} + 10958745600 a^{14} b^{31} c^2 + 731313623040 c^8 b^{31} a^{10} + 1175325465600 c^8 b^{31} a^8. \]

References


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